PRINT: ISSN 0975-1122 ONLINE: 2456-6322
Int J Edu Sci, 9(2): 151-161 (2015)
DOI: 10.31901/24566322.2015/09.02.07

Mathematics Education Prospective Teachers' Errors Patterns on Grade 12 Mathematics

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KEYWORDS Amateur Concepts. Algebra. Functions. Error Analysis and Diagnosis. Prospective Teachers

ABSTRACT This paper investigated and identified first-year Mathematics Education prospective teachers' errors patterns on Grade 12 algebra and functions. The research focussed on a sample of (n=63) first-year Bachelor of Education (BEd) Mathematics Education prospective teachers. The prospective teachers wrote mathematics tasks selected from past National Senior Certificate Examinations. Prominent errors that the prospective teachers still make on high school mathematics tasks were identified. The research found that most errors prospective teachers made were conceptual. Prospective teachers also showed that in the main they did not understand mathematical notation and terminology. The research recommends that error analysis be central to a bridging course for first-year Mathematics Education prospective teachers to develop conceptual understanding which was identified as the main source of errors in this report.

INTRODUCTION

Error analysis and diagnosis is a critical aspect of mathematics pedagogy (see Makonye and Hantibi 2014). Schulman (1986) argued that knowledge of students' errors and misconception in a discipline is a critical part of Pedagogical Content Knowledge (PCK). PCK is knowledge for teaching of a particular subject that is quite different from the subject matter itself or pedagogic knowledge per se. It concerns knowing ways of formulating and presenting a subject matter so that it is comprehensible to learners. Knowledge of learner errors and misconceptions is particularly important because when prospective teachers learn new mathematics concepts, they often invent alternative concepts quite different from the ones teachers expect them to learn (Nesher 1987; Smith et al. 1993; Makonye and Luneta 2013). This happens because of learners' failure in the adaptation process as they assimilate new knowledge into current schemas in a situation which actually calls for the recognition of the old schema or its complete replacement through accommodation. Knowledge of prospective teachers' alternative conceptions or misconceptions is important for teachers' PCK as it helps to devise strategies for learners to revise their concepts.

The research was undertaken to determine mathematics major student-teachers' error patterns on Grade 12 school mathematics. Drawing from the practice-based concept, this research identified problems in the workspace and sought to understand these problems in order to improve practice. In an educational setting, practice-based research aims to identify problems in educational practice, gather evidence on it and interpret the evidence in order to improve teaching and learning. Such research may lead to real change in education as it fills the gap between theory and practice. Practice-based research in education is centred on 'evidence-based' policy and practice (Ros and Vermeulen 2010). Practice-based or practitioner led research is a form of applied research. The Organisation for Economic Cooperation and Development (OECD) (2002) define applied research as, 'original investigation undertaken in order to acquire new knowledge, directed towards a specific practical aim or objective' (p.30). According to Ros and Vermeulen (2010), practice-based research is more helpful to effecting change in teaching and learning than many theoretically-based research papers which tend to be too academic and difficult to read and access. Practitioners find it difficult to access and apply such theory-based research in their teaching. Also such articles tend to have little practical recommendations.

It is argued in this paper that if prospective teachers' erratic mathematical thinking and reasoning is not unpacked for both teachers and learners' consideration, little progress in learning mathematics can be expected, because teaching may not be addressing learners' current difficulties. As learners' misconceptions are identified and resolved one by one, the learning of new mathematics concepts becomes easier for them. This is because mathematics is a hierarchical subject in which higher mathematical concepts can only be understood if lower ones have also been understood. Also, research suggests that a single misconception, a bug, can snowball to reproduce a cluster of errors along the way (Nesher 1987). The snowballing effect of misconceptions makes learning mathematics hugely difficult. A misconception effectively locks out the understanding new concepts, whereas understanding a concept ushers understanding of new concepts. In mathematics the success of understanding one concept leads to the success of understanding another and vice versa. Therefore knowledge of learners' errors associated with varied mathematical tasks is imperative for the progression of mathematics teaching and learning including in teacher education.

In South Africa, research on learner error analysis has been quite minimal. Although regular international standardized tests involving South Africa (see for example, Howie 2001; Moloi and Chetty 2011; Taylor and Taylor 2013) focused on performance in mathematics (and science), the results were mainly aimed at national performance comparisons. Very little data on the errors learners make is discussed. However, of late, some researchers are beginning to take this research field quite seriously (see for example; Luneta and Makonye 2012; Makonye and Luneta 2014). Luneta in particular regards error analysis researchin mathematics (and physics) imperative as it could leverage teachers' effectiveness to take learners' understanding of these subjects to a level not yet reached in the South African educational landscape given the documented low performance of South African learners in mathematics and science.

Literature on Patterns of Errors and Misconceptions in Functions and Algebra

There are various reasons why students make many errors and misconceptions in algebra and functions. According to Posamentier (1998), students embrace mini-theories about mathematical ideas which they use to interpret new mathematical concepts. Also, Egodawatte (2011) argued that errors and misconceptions in algebra emanate from learners' misunderstanding of the essence of algebraic variables as well as,under or overgeneralization of arithmetic laws to algebra. In addition, Fischbein and Barash (1993) stressed that students' intuitive ideas about mathematical concepts and procedures often become obstacles in learning algebra. They pointed out the misconceptions learners have in relating algebraic expressions like, $(c+b)^5$ and a^5+b^5; $3(a+b)^2$ and $3a^2+3b^2$. In most cases, learners held the misconceptions that these expressions were equal. The causefor this tended to be effortsto intuitively apply the distributive rule of multiplication over addition: $m(a+b)=ma+mb$ to these cases. The other intuitive errors are $\cos(a+b)=\cos a + \cos b$; and $\log(a+b) = \log a + \log b$ (Donaldson 1963). While in solving quadratic equations, Davis (1984) pointed out that when learners are taught to solve quadratic equations by first factorizing to the form $(x-a)(x-b)=0$, then x is equal to either a or b; learners go on to generalize that if $(x-2)(x-3)=5$; then $x=5+2$ or $5+3$, wherein they would have generalizedthat because 5 is a number just like 0, so they must handle the equation in the same way. Others are errors of need to have closure: $3+2x$ is seen as not closed so the answer becomes $5x$ (Artzt and Armour-Thomas 1992). This error is called the conjoining error.

On functions, students misunderstand the essence of functional representations (Makonye 2014). Some students think that on a speed-time graph (See Fig. 1), the vehicle is moving uphill

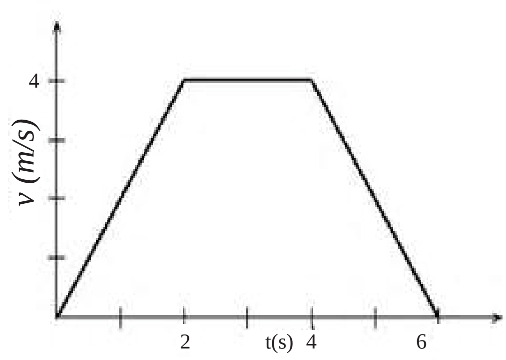


Fig. 1. Speed-Time functional graph (Insert here)

from O to A, then moves on flat ground from A to B, before moving downhill from B to C.

Misunderstanding functional and mathematical representations and symbolism are very common in mathematics, which is one reason why this paper focuses on these topics.

Objectives

This research aimed at understanding the pattern of errors and misconceptions that first-yearMathematics Education prospective teachers have on grade 12 school mathematics on the topics of functions and algebra. The research was guided by the following questions:

(a) What is the pattern of errors shown byprospective teachers in responses to Grade 12 algebra and functions Grade 12 mathematics tasks?

(b) What explanation can be availed for these patterns?

Significance

The School of Education program has embarked on improving our teaching through practice-based research.This is done under the theme, "we research what we teach; and teach what we research". Over the years we have realized the wide and compelling mathematics knowledge deficit of our student teachers. We have noted that most of our prospective teachers arrive at the university without the core mathematics competencies that they are supposed to have acquired at high school. Also during prospective teachers' teaching experience, these gaps in mathematics content knowledge often crop in the delivery of mathematics lessons. Such ignorance is embarrassing to the prospective teachers themselves and the learners they teach not to mention their university. In some schools, supervising teachers complain that our student teachers' content knowledge hardly measures up. Given this scenario, it is imperative that we address this problem so that the teachers whom we train, graduate with impeccable mathematics teaching qualifications. A research such as this one which focuseson student teachers' Grade 12 mathematics knowledge is important. It is important that once the research is done, prospective teachers' weaknesses can addressed by the time they graduate.

It is also seen in our experience that some teachers graduate from South African universities with gaps in Grade 12 level mathematics. Such teachers are not assets but liabilities who perpetuate sub-standard levels of mathematics teaching and learning in this country. In that event, the society has the right to point fingers at us, as the mathematics educators having failed them.

Theoretical Framework and Conceptual Framework

First, the research draws from Davis' (1984) ideas that liken the working of learners' minds to computer information processing systems. According to Davis, the human mind systematically operates in certain ways. Davis advocated that knowledge representation in a learners' mind is known as a frame. One of the laws that govern frames is the Brown-Matz-van Lehn Law, (Davis1984) which stipulates that top-level programs must run to their ultimate conclusions. In this case a learner has an overarching, very important procedure that governs the solution of a mathematics task. This higher procedure may require lower procedures to provide inputs into it for it to operate to its logical end. If these lower procedures are not available, accommodations are made, so that default procedures can be constructed and the higher level programs do not stop but can be completed. The default input programs may be erratic. So in the super-sub procedure interface, when obstacles are met, modifications are made to enablethe processing to continue. Davis asserts that the super- or sub-procedures invariably have a valid mathematical basis where they have been used productively in the past but may be wrongly retrieved in situations where they are not called for. This results in errors inthe learners' productions.

Davis (1984) also maintained the Feigenbaum Minimal-Discrimination Rule (FMDR), which states that information processing systems will make essential discriminations but may not make discriminations finer than necessary, that is to say discriminations are not made where they are presently not needed. As an example, he illustrated the primary-grade undifferentiated binary-operation frame. He explained that since learners first learn addition at school, when learners are asked to use other operations in their school lives, they ignore other operation signs and always add. They are fixated to the addition operation, the operation they first encountered at school. Davis claimed that frame retrieval may be cued by brief, explicit, specific cues or situational similarity.The frames, micro worlds and scripts can be compared to concept images (Tall and Vinner 1981; Vinner and Dreyfus 1989).

Davis reported on the characteristics of frames. Firstly, they serve as assimilation schemas for organizing input data. Secondly, he argued that errors reveal the inner workings and functioning of frames. Thirdly, frames all have correct earlier learning. Fourthly, each frame demands certain input information to be provided; if not available, default information is inputted. Davis argued that the persistence of frames is evidenced by the fact that frames had identical functioning in a variety of situations. Frames also seem to have a hold on people using them, and people go to great extends to protect their cherished frames. This is evitable by individuals being prepared to go to bizarre distortions to avoid challenges to the content of their frames. Davis also pointed that any mismatches to the frame results in modification of input data rather than the frame itself. To mathematics educators, Davis has no good news as he asserted that, "instruction often avails little against frames" (p.123). He documented that, "even if instruction produces a change in frame, this change is often not permanent; before long the frame reasserts itself and behavior reverts back to what it was before instruction" (p. 125). Piaget (1968) highlighted that the reversion to earlier wrong frames was due to cognitive developmental immaturities.

Davis explained that the creation and operation of frames follows orderly rules, rule of initial overgeneralization, rule of the FMD, that is discriminations are not made where they are not presently needed, top-level programs must run and contradictory semantic information (Davis and McKnight 1980) is frequently not influential in modifying algorithmic behavior.

This framework further presumes that errors and misconceptions can be understood in terms of imperfectly constructed concept images. Tall and Vinner (1981) describe the *concept image* as, "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes activated at a particular time when the concept image is evoked" (p.152).They contrast the concept image to the *concept definition* which is the, expression in words of the concept as it is formally held by the mathematical community. It is presumed that errors and misconceptions in mathematics occur if learners' concept images are at variance with concept definitions. Errors and misconceptions are aggravated and perpetuated if the learner's attempt to incorporate new concepts to inadequate or defective concept images.

In reconciling these frameworks, the researchers maintain that learners have errors and misconceptions as they refer new mathematical concepts to past conceptual frames, which they are not prepared to let go. Their misconceptions are also due to concept images that are the sum of the old frames and the new mathematical contexts they encounter. The frames are bolstered by the FMDR and the Brown-Matz-van Lehn Law that compels them to complete procedures even when they use inputs they are not sure of. In their justified maintenance of their conceptual frames and concept images, they fail to learn mathematics as they always look back instead of forward to the concept definitions provided by their teachers and other learning resources. Errors and misconceptions are aggravated and perpetuated if learners incorporate new knowledge into inadequate and defective concept images.

Davis' theory can be linked to Tall and Vinner's concept image and concept definition. Essentially the higher and lower procedures of working out mathematical tasks that Davis refers to can be compared to Tall and Vinner's in the sense that, when the learner acquires the concept which is aligned to the appropriate concept image that he has, the learner will operate at the higher procedure mode. Davis' frames seem to be static whereas concept images are dynamic.

Conceptual Framework

Drawing from Nolting's (1997) work, the errors in this paper were classified and aligned to Nolting's framework. The description below provides details of how the errors were categorized. The prospective teachers' errors were analysed with this conceptual framework.

Conceptual Errors

Conceptual errors are due to students' failure to reason or operate in settings involving the careful application of concept definitions, relations or representations.

Examples:

0.214 > 0.3 3+2x = 5x; f(x+2) = f(x) + f(2)

$\frac{d}{dx}[(3x+1)^2] = [\frac{d}{dx}(3x+1)]^2$

Procedural Errors

Procedural errors occur when there is failure to use *algorithms*: the steps to follow a rule to obtain a correct answer.

Example:

	T	U
	t	6
+		z

giving answers of 9; from 1+6+2 or 38;
from 6+2=8; and 1+2=3 since 1 has no pair so 2 becomes the default evaluation.

Application Errors

Application errors occur when one has sufficient conceptual or procedural knowledge but fails to apply them to solve a task

Example: $x + x = x^2$

Careless Errors

Careless errors are errors of performance due to sloppiness or tiredness or some other interference. They differ from the above three errors which are errors of competency.

MATERIAL AND METHODS

This exploratory ongoing paper randomly selected the prospective teachers with the expectation of gaining in-depth and sufficient information about prospective teachers' patterns of misconceptions and errors displayed when working on school mathematics tasks. The sixty-three (63) prospective teachers were drawn from first-year Mathematics Education prospective teachers in the Bachelor of Education programme. The data collection techniques utilized were document analysis and interviews. The preliminary analysis discussed in this paper draws on data from the analysis of prospective teachers'responses to algebra and functions tasks adapted from the 2011 Grade 12 National Senior Certificate paper 1 and 2.

Thesetwo papers were distributed to the mathematics prospective teachers. No examination restrictions were applied when the prospective teachers wrote the papers. The researchers intended to solicit as much data as possible and as such prospective teachers were allowed to complete the papers at their own leisure. A return rate of fifty-eight percent was realized.The papers covered the five broad areas of the mathematics curriculum, namely, algebra, functions, sequences and series, trigonometry and analytical geometry. Mathematics questions vary with respect to content and the cognitive processes requires in finding solutions to the questions. The implication is that different cognitive demands are likely to induce different kinds of responses. The research employed the Stein et al. (2000) mathematics task analysis framework to analyse the cognitive demand of the questions. The questions were classified as low level questions categorized as procedure without connections according to the Stein et al. (2000) Mathematical Task Analysis Guide. The tasks required use of procedures requiring no explanations and limited cognitive demand for successful completion. Ten prospective teachers were selected to probe for their error patterns and misconceptions in answering the functions and algebra questions.

RESULTS AND DISCUSSION

Table 1 presents a description of the items that the prospective teachers responded to in paper 2. Item 1 elicited knowledge of algebraic concepts of equations, inequalities and surds. Item 5 featured on functions concepts of equation of asymptote, intercepts, sketching of graphs and finding the range of a function.

Content analysis was utilized to examine the responses on the scripts. Three lenses were employed to analyse the data: a broad analysis of attempt and non-attempt of items, analysis of type of responses, and analysis of misconceptions and errors. Two broad categories of response and non-response were identified. The response category refers to an attempt to answer the item whereas non-response referred to no attempt on the item. Table 2 shows the prospective teachers'attempts and non-attempts of the itemsusing percentage frequencies.

Table 2 reflects that item 1.1.1 was attempted by all the prospective teachers compared to

Table 1: Items descriptions

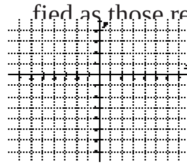
Question		*Topic*
Item 1	Solve for x, correct to TWO decimal places, where necessary:	Equations
	1.1.1 *(x(x-1) =12*	
	1.1.2 *(7x²+18x-9>0*	Inequalities
	1.2 Simplify completely, without the use of a calculator:	Surds
	$\left(\sqrt[5]{\sqrt{35}+\sqrt{3}}\right)\left(\sqrt[5]{\sqrt{35}-\sqrt{3}}\right)$	
Item 5	Consider the function $f(x)=\frac{3}{x-1}-2.$	
	5.1Write down the equations of the asymptotes of f.	Equation of asymptote
	5.2 Calculate the intercepts of the graph of f with the axes.	Intercepts
	5.3 Sketch the graph of f on the grid below.	Sketching graphs
	5.4 Write down the range of y $=-f(x).$	Range of a function

Table 2: Analysis of responses and non-responses

		Attempts (%)	*Non-attempts (%)*
Item 1 (Algebra)	1.1.1	100	0
	1.1.2	87	13
	1.1.3	61	29
Item 5 (Functions)	5.1	65	35
	5.2	61	29
	5.3	57	43
	5.4	70	30

item 1.1.3 which was attempted by sixty-one percent of the prospective teachers indicating that twenty nine percent did not attempt the question. All the sixty-three percent of the prospective teachers attempted the equation item but thirteen percent failed to attempt the question on inequalities as much as they did not attempt the question on surds. Item 5 comprised of questions on functions. At least sixty percent of the prospective teachers attempted the items although that cannot be said for item 5.3 where forty-three percent did not attempt to sketch the hyperbola with seventy percent giving the range of the function. Overall, more responses were realized than non-responses.

A deeper examination was conducted to reveal the nature of the response of the attempted items (see Table 3). Categories of responses were developed and characterized as, correct, partial, and incorrect. The correct response indicated that the participant's solution was correct. The strategy for the correct method was not considered at this stage. Partial responses were identified as those responses that were incomplete re-



ther correct or incorrect. The par- ad some elements of correct pro- ıld have lead to an accurate re- ct responses were identified as nents that resulted with incorrect solutions.Table 3 provides examples of responses according to the categories of responses:

The focus of the investigation was to identify the types of errors and misconceptions displayed by the prospective teachers. A further examination of the responses was crucial in identifying the errors and misconceptions. The categories of the responses provided an understanding of the nature of responses. Table 4 presents the responses as classified in four categories: correct, partial, incorrect and non-response.

For instance, although item 1.1.1 was attempted by all the prospective teachers, seventy percent correctly answered the item with seventeen percent incorrect and thirteen percent partial responses. Item 5.4 was attempted by seventy percent with nine percent of these prospective teachers successfully giving the range of the function and fifty-seven percent unable to doso. Item 1.1.2 was attempted by eighty-seven percent of the prospective teachers but only seventeen percent correctly answered the item. The

Table 3: Nature of responses to items

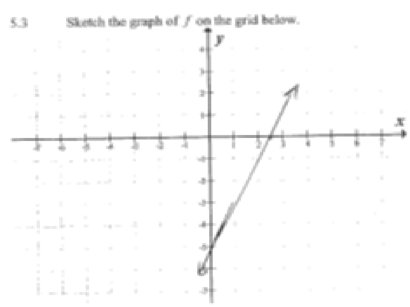
Correct Response	1.2 Simplify completely, without the use of a calculator: *Responses By Teacher A* $= \sqrt{35 - \sqrt{35},\ \sqrt{3} + \sqrt{35},\ \sqrt{3} - 3}$ $= \sqrt[5]{35 - 3}$ $= 32^{\frac{1}{5}}$ $= 2$	5.2 Calculate the intercepts of the graph of *f* with the axes. *Responses By Teacher B* y-intercept: x = 0 x-intercept: y = 0 $y = \frac{3}{0 - 1} -2$ $0 = \frac{3}{x - 1} -2$ $y = \frac{3}{-1} -2$ $+2 = \frac{3}{x - 1}$ $3 = +2x - 2$ $y = = -3 -2$ $\frac{5}{2} = \frac{2x}{2}$ $y = -5$ $\frac{5}{2} = x$
Partial Response	1.1.2 Solve for x, correct to TWO decimal places, where necessary: $7x^2 + 18x - 9 > 0$ *Response by Teacher C* *(7x-3)(x+3)>0* *7x 3 OR x -3* *x 3/7*	5.3 Sketch the graph of f on the grid below. *Response by Teacher F*
Incorrect Responses	1.2 Simplify completely, without the use of a calculator: $\left(\sqrt[5]{\sqrt{35} + \sqrt{3}}\right)$ $\left(\sqrt[5]{\sqrt{35} + \sqrt{3}}\right)\left(\sqrt[5]{\sqrt{35} - \sqrt{3}}\right)$ *Response by Teacher E* $= \sqrt{35} + \sqrt{3}$ $= 5$	

Table 4: Analysis of item 1 and 5 responses

		Categories			
		Correct (%)	*Partial (%)*	*Incorrect (%)*	*Non-response (%)*
Item 1	1.1.1	70	13	17	0
	1.1.2	17	49	21	13
	1.1.3	17	0	44	39
Item 5	5.1	57	4	4	35
	5.2	31	26	4	39
	5.3	14	4	39	43
	5.4	9	4	57	30

researchers attribute this to lack of understanding of the use of the inequality sign. The prospective teachers failed to establish the meaning of the inequality symbols suggesting that they could not create meanings for rules and procedures that govern actions on the inequality symbol. One prospective teacher expressed that;

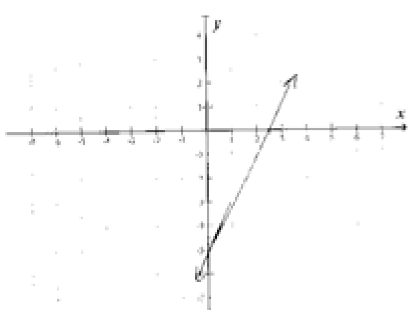
I don't have problems solving equations but then… then with inequalities, I can solve the left side then…ahh… I don't know whatto do with the sign.

This response confirms that in general, the prospective teachers can work with equations but struggle with inequalities as reflected in Table 4.It is a practice in South Africa that inequalities are taught as a subordinate subject of equations. Boero and Bazzini (2004) suggest that this approach "trivializes" inequalities, resulting with emphasis on "routine procedures, which are not easy for students to understand, interpret and control" (p.140). The prospective teachers performed poorly on inequalities, surds, sketching hyperbolic functions and finding the range of such functions.

The intention of this investigation was to identify the errors and misconceptions displayed in responding to the mathematics paper. To do this, the researchersconcluded that errors could be located among the partial and incorrect

Table 5: Exemplar errors

Conceptual Errors
Response to Question 5.3



Application Errors
Response Question 1.1.2

$(x = 3)\ (7x - 3) > 0$

$-3 < x < \frac{3}{7}$

Procedural Errors
Response to Question 1.2

$\left(\sqrt[5]{\sqrt{35}+\sqrt{3}}\right)\left(\sqrt[5]{\sqrt{35}-\sqrt{3}}\right)$

$= \sqrt[5]{35} - \sqrt{3}$

$= 5$

Unintended Errors
Response Question 5.2 .5

$\frac{3}{x - 1} - \frac{2}{1} = 0$

$\frac{3}{x - 1} = 2$

$2x - 2 = 3$

$2x - 5 = 0$

$x = \frac{5}{2}$

$y = \frac{3}{\frac{5}{2}} - 2$

$= \frac{5}{2} \times 3 - 2$

Incomplete Correct Errors
Response Question 1.1.2

(7x - 3) (x + 3) > 0

7x 3 OR x -3

x 3/7

responses. The types of errors were characterized as, conceptual, procedural, application and unintended errors using the classification by Nolting (1998). While many authors have provided several classifications of errors (see for example, Donaldson 1963; Hirst 2003), the researchers in this study preferred the Nolting's (1998) categories of errors. Reference is made to Table 5 which provides examples of responses according to the categories errors produced. According to Nolting (1998), conceptual errors are those made when learners do not understand the mathematical properties or principles required to successful answer the tasks. The response to question 5.3 indicates that the prospective teacher lacks understanding of transforming a verbal description of a function to a graphical representation. A random function that has graph that does not depict the verbal description was drawn, indicating lack of knowledge of hyperbolic functions. Procedural errors are made due to prospective teachers' misuse of a rule, formula or algorithm when carrying out a mathematical calculation. The response classified as a procedural error in Table 5 strongly suggests that the prospective teacher made an error in following the procedures for working with binomial products. The first step in the solution indicates that the teacher attempts to find the product of the two number expressions but makes an error of judgement.

An application error occurs when a prospective teacher understands the concepts but cannot apply them to a specific problem situation. The example shows that the prospective teacher has knowledge of solving inequalities but fails to represent the solution on the number line.A careless or unintended error occurs when the prospective teachers have the required knowledge to perform a task correctly but due to some interference or distraction, the prospective teacher makes errors. The example given of question 5.2 for this category shows that the prospective teacher had the required knowledge to find the intercepts of the function but somehow was unable to complete the process. However, another category "incomplete response with correct statements" was included. A response to question 1.1.2 is given as an example of an incomplete correct error. The response demonstrates that the prospective teacher has some knowledge of the procedure for the expansion of binomial products but does not understand the use of symbols in the expansion process. The teacher prefers to leave the solution statements incomplete.

Table 6 presents the summary of the error classifications using the analytical framework for identifying the errors from the partial and incorrect responses. Table 6 shows that conceptual errors (39%) are more prominent than other errors. There is an indication that prospective teachers do not understand the mathematical properties or principles required to successfully answer the tasks. This is particularly clear that more conceptual errors were realised in responding to the functions task. Of great interest is that application errors correlate with procedural errors. The application and procedural errors by prospective teachers were due to weak competence in algebraic processes such as the notions of a variable and substitution.

Prospective teachers also had errors due to mathematical symbolism and terminology. Difficulty with the conceptualization of the symbolic representation impacts the understanding of the function concept. It indicates that the prospective teachers who did not possess sufficient

Table 6: Analysis of errors

		Errors				
		Conceptual	*Application*	*Procedural*	*Unintended*	*Correct/ Incomplete*
Item 1	1.1.1	28	14	14	44	0
	1.1.2	6	31	25	19	19
	1.2	30	30	30	10	0
Item 5	5.1	50	0	0	50	0
	5.2	28	0	14	28	28
	5.3	50	40	0	10	0
	5.4	58	21	0	7	14
Overall Type of Errors Committed		39%	21%	13%	26%	1%

knowledge of the function concept were bound to make application errors. What is depicted by Question 5.3 is that the prospective teachers could not shift from one functional representation to the other. Mathematical connections between concepts are induced when different systems of representation are promoted. This is in line with Duval's (1999) suggestion that mathematics activities should provide opportunity for "the construction of a cognitive structure by which the prospective teachers can recognize the same object through different representations" (p.12).The task required the prospective teachers to relate the visualization process through the transfer of symbolic objects by processing and representing these objects to a visual representation. Clearly, this process requires a transformation of a perceptual apprehension of the symbols to a sequential apprehension of the concepts. Although fifty-seven percent correctly identified the asymptote (question 5.1) only fourteen percent could sketch the graph (question 5.3), suggesting that they were bound to make conceptual errors and application errors. As mentioned, unintended errors occurred when the learners have the required knowledge to perform a task correctly but due to some interference or distraction, the learner makes errors.The greatest unintended errors occurred for questions that had the highest correct response rate.

It is worthy to note that prospective teachers have concept images (Vinner and Dreyfus 1989)of algebra and functions which are different from what they have been taught. In their way to construct mathematical concepts, they often resort to cognitive representation systems (Davis 1984) inadequate for the considerably high level mathematics that was required in thesetasks. Prospective teachers tried to assimilate new mathematical ideas into the present weak cognitive structures they had, instead of building new ones. The patterns of errors displayed in this paper are an accumulation of prospective teachers'cognitive problems with school mathematics over many years that remain unaddressed. The prospective teachers have carried misunderstanding of mathematics ideas and procedures over many years culminating in fuzzy mathematical ideas they showed in this paper. The lack of conceptual understanding effectively blocks prospective teachers' epistemic access and progression in mathematics at all levels.

CONCLUSION

This research identified patterns of errors prospective teachers made in their responses to various mathematics problems basing on an argument that errors are an indication of some misconception. A misconception indicates a misunderstanding of a concept and an error indicates a misapplication of a concept. The research found out that most errors prospective teachersmade were due to weak competence in algebraic processes including the notions of a variable and substitution. Prospective teachers also had many errors due to mathematical symbolism and terminology.The findings indicate that conceptual errors were more prominent than the application, procedural and unintended errors, clearly suggesting that the mathematical proficiency level was low. Although the prospective teachers know how to use the procedures they could still not apply these procedures to specific situation and thus the occurrence of a substantive number of application errors.

RECOMMENDATIONS

In view of this research on the patterns of errors of student teachers on algebra and functions tasks, we recommend that:

- all grade 12 mathematics be re-taught for the first semester in year 1 of the mathematics major prospective teachers Bachelor of Education course. This re-teaching must take into consideration the need to strengthen conceptual understanding byusing multiple representations and models.
- in the implementation of the course, lecturers take cognizance of the errors and misconceptions prospective teachers have so that teaching is directed at these epistemic obstacles.

ACKNOWLEDGEMENTS

This paper is written under the auspices of the UNESCO Chair in Teacher Education for Diversity and Development at the University of Witwatersrand for which the researchers would like to express their appreciation for support in conducting this research.

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