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Teaching Functions Using a Realistic Mathematics Education Approach: A Theoretical Perspective

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ABSTRACT This paper discusses how the notion of a function can be developed for high school learners through the use of different representations and models; verbal, visual, graphical and symbolic. The realistic mathematics education approach provides the framework that guides the discussion. Using the matchstick problem as an example, multiple-representations of the function concept inherent in it are mathematised. The possible representations of the function are; geometric patterns, independent - dependent variables, ordered pairs, fish diagrams, number sequences (quadratic sequences, arithmetic sequences, geometric sequences), dual bar graphs, graphs on the Cartesian plane and the functional f(x) symbolism. It is argued that multiple representations that start with the informal and every day and then gradually progress to the formal and abstract, help learners to gain insight of the big idea functions in mathematics. The paper provides mathematics educators a platform which facilitates a realistic mathematics education approach to teaching functions which can be extended to other mathematical topics.

INTRODUCTION

Many times students learn mathematical concepts without understanding them because some teachers teach them procedures without connections to their contextual experiences (Stein et al. 2000; Balacheff and Gaudin 2010; King and Bay-Williams 2014). Such teachers, often called traditional teachers, hold instructional conceptions that are usually shaped by the methods which they used to master mathematical concepts and the teaching methods used by their teachers during school days when they were learners themselves (Lubisi 1997; Viirman 2012). The primary goal of traditional teachers is for learners to find answers to problems using legitimate mathematical processes such as applying formulae, definitions, axioms or theorems. This approach to teaching mathematics that emphasizes procedural competencies is often done at the expense of learner conceptual understanding of mathematical concepts and procedures (Hodgen et al. 2009). When teaching mathematical functions, traditional teachers begin with a definition, that it is an equation involving inputs that lead to unique outputs with the property that each input is related to exactly one output, rather than the learners given situation definition in which they can form the functions themselves. When teaching functions to Grade 9 learners in South Africa or form 2 learners in Zimbabwe, such traditional teachers often advise learners that in order to determine whether a graph represents a function, or not, they need to use the vertical line test. The test asserts that if the vertical line crosses a graph once only, then the graph is deemed a function and if it crosses more than once then it is not a function. Also learners may be asked to substitute an integer in a given function f(x), (say at $x = -2$) to find if there is a single real number output. Although learners may perform this test by making correct substitutions to evaluate f(-2), they hardly understand the essence of the notion of the function and how important it is to mathematical discourse. Such learning is problematic because it encourages learners to memorize procedures without understanding them which may limit their transfer or connections of learnt concepts in new and novel situations. Such learning is restrictive and may not enable learners to achieve the five fundamental goals of learning mathematics; of reasoning, making connections, applications, communication and problem-solving (National Council of Teachers of Mathematics 2014).

Furthermore, learning mathematical concepts by rote where learners rarely understand why they are bothered to study concepts such as functions can de-motivate them. In the absence of contextual learning of functions learners can find it difficult to justify why they ever learn them. Yet the function concept is one of the most fundamental concepts and biggest ideas of modern mathematics that forms the glue that ties mathematical concepts together (Shenitzer and Stillwell 2002; Viirman 2012). The argument of this paper is that in line with reform pedagogy emphasising constructivist learning, "teaching mathematics is equipping students with conceptual understanding of the process skills that enable students to individually or collectively develop a repertoire for constructing powerful mathematical constructions that concur with viable mathematical knowledge" (Nyaumwe 2004: 25). Thus, the function concept being such a fundamental concept in mathematics deserve learner active participation for them to conceptually understand it and be able to apply it in their future mathematical studies. Traditionally, mathematical procedures were given more importance in the teaching and of learning mathematics, while conceptual understanding if at all was given little attention. Such an approach flies in the face of learner conceptual understanding which really has potential to enhance learning of mathematics in the long term (National Council of Teachers of Mathematics 2014). Active construction of mathematical concepts is possible because from a constructivist perspective, mathematical concepts are tentative, intuitive, subjective, and dynamic that they are contextual as they originate from human activities within a given context, and therefore, are not universal and fixed. This notion gives insight to believe that Realistic Mathematics Education (RME) (Freudenthal 1991; Gravemeijer 1994; Webb 2010) approaches have potential to help learners to understand the big ideas of mathematical concepts such as functions.

In efforts to understand the active nature of teaching the function concept to Grade 9 learners the present study had two goals to pursue. Firstly, the study seeks to demonstrate the pervasiveness of the function concept in mathematics where it is often denoted by an unconnected variety of mathematical representations. Secondly, the study attempts to demonstrate how the realistic mathematics education approach can help learners to construct themselves complete understanding of the mathematical structure of functions.

The significance of the study lies in the assertion that the function concept is one of the biggest ideas that builds the discipline of mathematics as it is the substance that keeps together the underlying mathematical concepts and procedures. Despite this fact, the function concept is often misunderstood, by teachers and learners. For example, the basic arithmetic operations of addition, subtraction, multiplication and division are functions in that each of them process two numbers in a certain way to get another number. Thus all arithmetic operations can be viewed as functions that map points on the Cartesian plane R^2 to the real line R^1. The appreciation that the processes of addition, subtraction, multiplication and division represent the object called a function is not commonly understood. This unawareness of the fundamental laws governing mathematics particularly by teachers is regrettable since their teaching often marginalises the core concepts of mathematics. This paper is important because it brings to light some assumptions held by both teachers and learners on the function concept that are seldom considered in teaching and learning mathematics.

The function concept is often taught without relation to everyday context. Formal mathematical symbolism of the concept such as f(x) is sometimes prematurely introduced to learners which may result in some learners developing misconceptions about it as the manner of teaching is divorced from meaning to the learner. When the function concept is wholly taught using the mathematical context that learners do not understand, the learners may face find difficulties to understand it. The paper delves deep into the minute details of the function concept which may inform teachers about how to introduce the concept to learners in a conceptual way. The argument of the paper rests on the belief that some teachers are not aware of the obvious notion that number sequence is also a function, yet they struggle to get examples to illustrate the notion of a function

The argument of the paper rests on the assumption that the use of the realistic mathematics education approach may provide learners with examples from their day- to- day experiences, which may help to counter their lack of understanding of the function concept. The approach may also induce interest in the learners to learn mathematics as it is assumed that realistic contexts may capture their affective domain. The approach may facilitate learners' readiness to accept mathematical symbols on function concept when they are eventually introduced because learners may have seen the necessity for the symbols. The realistic mathematics education approach is also very useful in contextualising the same concept such as a function in different forms. This multiple modelling of concepts may help learners to understand the mathematical concepts associated with the function concept in their totality. By paying attention to detail on the obvious routine events, this theoretical paper may be significant in advancing the quality of teaching and learning of mathematics in general and the topic functions in particular.

Literature Review

The realistic mathematics education approach is an important approach used to teach mathematics in many countries such as the Netherlands, the UK and the USA (see for example, Lange 1996). Yet this approach has not been fully exploited to teach mathematics in Africa, particularly in South Africa which has a low learner achievement rate in mathematics. Traditional approaches are based mainly on exposition of a mathematics problem that is solved through a model answer on blackboard. Such "chalk and talk" approaches which characterise most classrooms can work for some learners, but most learners may come to hate mathematics through this method. This is because learners do not know the reason for learning mathematics other than to pass examinations. Other learners are perplexed by mathematical notation which characterizes mathematics. The result is that for most learners mathematics is rendered a dull, lifeless and meaningless subject. In many cases, learners develop various misconceptions in mathematics because they are expected to memorise too many mathematical concepts that are not relevant. The realistic mathematics education approach addresses this problem by encouraging mathematics to be more relevant and appealing to learner needs. RME makes learning mathematics meaningful and enjoyable to learners. Once learning is meaningful and enjoyable to learners, the sky is the limit for their successes.

The philosophy of realistic mathematics education Realistic Mathematics Education (RME) holds mathematics as a human activity that is connected to reality (Treffers 1991). This theory, originated from Freudenthal's (1991), argues that mathematics teaching should always be realistic; connected to real situations, have connections in learners' everyday lives and be relevant to society at large in order to be of human value. The word realistic goes beyond that mathematical problems originate from the learners' real world contexts. It also includes mathematical problems encountered in mathematics learning as long as they are relevant. In line with consructivist-fallibilist philosophy (Hersh 1997), Freudenthal regarded mathematics as a human activity that should not be studied only for its aesthetical beauty as in Platonic and Absolutist philosophy of mathematics (Ernest 1991). He argued that mathematics should be studied for its utilitarian purposes because at the very beginning, mathematics in classical times (such as during ancient Egypt and Babylonia), was invented as a tool for solving the practical problems that humans encountered in their daily lives. Such problems involved counting, measurement ofland and time. Such mathematical processes were necessitated by the need, for example, to allocate agricultural land fairly along the fertile river valleys where civilisation started. Calendar and time keeping; aided by careful studies of the regular movement of heavenly bodies; such as planets and stars, were important for interpreting planting and reaping seasons, as well as keeping religious ceremonies.

Thus it can be so argued that mathematics was invented by humans to serve their purposes. Being a human activity, it requires a human element in its learning. As such, it has to be appealing to the curiosity of those who learn it. Also, RME argues that mathematics ought to be taught through guided reinvention using carefully chosen realistic problems where students "can experience a similar process compared to the process by which mathematics was invented" (Zulkardi 2002: 4). The reinvention being underlined by exploration, trial and error, intuitive and conjecturing approaches in which learners are encouraged to engage in mathematical discourse related to their everyday experiences. Freudenthal (1991) proposed that the process of doing mathematics is mathematisation.

Two types of mathematisations which were formulated explicitly in an educational context by Treffers (1991) are horizontal and vertical mathematisation. In horizontal mathematization, the students come up with mathematical tools which can help to organize and solve a problem located in a real-life situation. On the other hand, vertical mathematization is the process of reorganization within the mathematical system itself. Freudenthal (1991) and Gravemeijer (1994) argued that horizontal mathematisation involves going from the world of life into the world of symbols, while vertical mathematization means moving within the world of symbols from simplex to complex.

RME is a teaching and learning theory that views mathematics as a human activity that is connected to reality (Treffers 1991). Thus, the reform curriculum emphasise problem-solving as a teaching approach that enables learners to develop mathematical concepts from common activities in their environment through solving problems that they encounter in their environments. Mathematical problem-solving generally involves presenting learners with written word problems in which the learners interpret the problems, and devise methods to solve them, following certain mathematical procedures to obtain a result. Problem-solving activities permeate the boundaries of the subjects offered in a curriculum as they provide holistic contexts in which mathematical concepts and skills can be developed and mastered (National Council of Teachers of Mathematics 1989). The advantage of problem-solving in a realistic environment of teaching mathematics is that it can bridge informal mathematics and formal mathematics.

The major assumption underpinning this study was that the realistic approach holds greater potential than the traditional approach for developing learners' conceptual and procedural knowledge on the notion of a function (Hiebert and Levevre 1986). When developed in a realistic way, the mathematical notion of a function can be modelled in various ways such as verbally, numerically, geometrically, algebraically, graphically (Suh 2007). These multiple-representations of the function concept though apparently different at face value actually convey the same concept. A unique underlying mathematical structure of the function, relates to all as these representations which are isomorphic to each other. For example the representations; geometric pattern such as matchstick arrangements, number sequence, algebraic n^{th} term and graph represent what is called a discontinuous function (verbal form).

As the RME approach argues, learners must be guided to re-discover mathematics by first working with motivating contexts. As such, it is crucial for teachers to select mathematical problems that trigger learners' interest. These must intrinsically invite learners to want to engage with the problem and want to solve it. Learners attack the problem with all the prior knowledge that they already possess and as well as any resources available to them. The RME approach contrasts with the top-down teaching approach in which teachers first expose the mathematical knowledge and procedures. Learners are forced to learn them without the *"raison de étre"* (a French phrase which means *reason for existence*) to learn them. In contrast, in the realistic approach, learners begin to attain formal mathematical knowledge and symbolism when they understand the need for them. They appreciate them because they come to notice their brevity and elegance. Also should they forget a mathematical formula, they can easily retrace their processes when they first constructed the formula. So when they use the realistic approach, learners consciously seek for structure, for mathematical symbols and terminology for the concepts that they have already handled. Mathematical procedures and knowledge are thus reinvented and rediscovered in a meaningful manner. Learners then may be able to easily vertically mathematise as they will be using mathematical knowledge whose basis they are comfortable with.

In working with the realistic contexts learners may start by using intuitive and informal methods to explore the mathematical problem at hand. As such, trial and error is necessary "bootstrapping" in which learners try to figure out the exact nature of the problem and its parameters. They figure out how the problem relates to what they understand and know. At this level learners are encouraged to freely formulate and test conjectures on the problem. As learners move forwards and backwards with their attempted solutions to the problem, they begin to understand its characteristics, as well as viable and unviable solution paths. Freudenthal (1973) and Treffers (1991) encourage moving back and forth from informal mathematical methods to precise and formal mathematics during the process of mathematising.

In the following paragraphs, we discuss how the study of manipulatives and visuals (in the matchstick problem) can help learners to deeply understand the different informal representationsof a function alluded above, through horizontal mathematisation. This leads to the need for the formal mathematical structure and symbolism of the function.

Matchstick Problem as the Basis for Teaching Different Functions Representations

An illustration of how the mathematical function concept can be developed using matchsticks in learners' environments is discussed next. Matchsticks are usually available the environments of many learners. The match sticks can be used to make shapes as illustrated below.

Matchsticks are arranged to form of successive squares by adding matchsticks to a previous diagram as illustrated in the Figure 1.

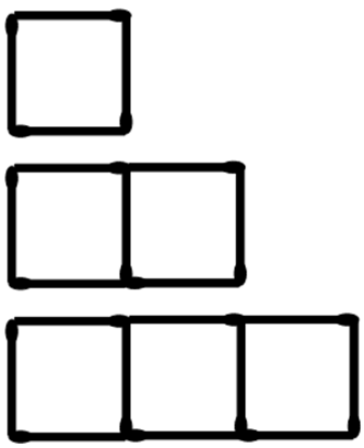

Fig. 1: Matchstick pattern

From the squares in Figure 1 learners are asked to find the number of matchsticks in each of the diagrams. The learners will notice that the first diagram has 4 matchsticks, the second has 7 matchsticks, the third has 10 matchsticks and the fourth has 13 matchsticks. Learners may be asked to answer the following questions in order to extend their thinking.

i. Hence, find the number of matchsticks required to make 55 squares
ii. Find the number of squares which can be made by 226 matchsticks
iii. Draw the graph to represent the relationship between the number of squares and the number of matchsticks.

In answering this question it is important for learners to realise that to get the next figure in the pattern three matchsticks are added to the previous one. This is the first aspect of informal mathematising; horizontal mathematisation. Figure 1 gives a pictorial representation of the pattern that emerges from arranging the match sticks

Table 1 is used to systematically analyse the pattern inherent in the matchstick visuals; which pattern at first is hidden from learners. Drawing a picture and drawing table are some of the heuristics suggested by Polya (1973) for problem solving.

Figure 2 below prepares learners for abstract reasoning as they need to figure out the colours for shapes that require large numbers of squares, for example 55 squares. Table 1 shows how learners can be lead to move away their thinking from concrete objects to abstract thinking that gives rise to a function concept.

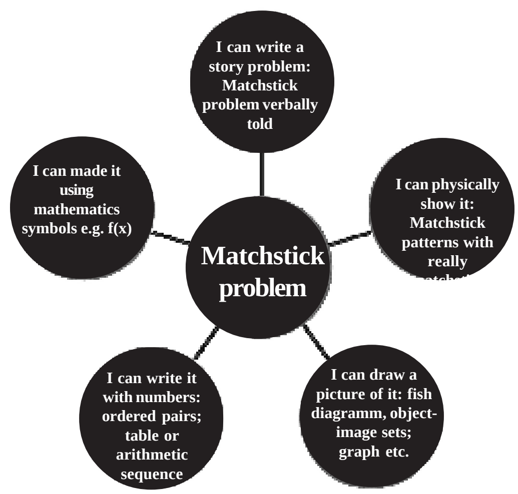

Fig. 2. Modelling the matchstick problem in different ways and levels of mathematisation

From Table 1 the function suggested by the matchstick problem is f(x) = 3x +1, where x is the number of squares and f(x) the number of matchsticks in that square.

Multiple Representation of the Function Concept Represented by the Matchstick Problem

The function of the matchstick problem can also be represented in many other ways such as using a fish diagram. The Fish diagrams can illustrate the input output definition of a function

Table 1: Conjecturing and inducting the pattern/function inherent in the visual

No. of squares	*1*	*2*	*3*	*4*	*5 … n -1*	*N*
No. of matchsticks (Simple counting thematchsticks in the real Fig. 1)	4	7	10	13	16	
No. of matchsticks: Conjecturing (Level 1) To get the next figure We just add three more matchsticks to the matchsticks in the previous figure (shown in different colour in Fig. 2)	4	4 + 3	7 + 3	10 + 3	13+3 (n-2)+3	n -1 + 3= = n-1+3 = n+3 -1 = n+2
No. of matchsticks: Conjecturing (Level 2)	4	4+3	4+3+3	4+3+3 +3	4+3 +3 +3 +3	4 + a certain number of threes (the number of threes which we are not sure of)
No. of matchsticks: Conjecturing in words (Level 3)	4 plus zero three	4 plus one three	4 plus two threes	4 plus three threes	4 plus four threes	
No. of matchsticks: Conjecturing in arithmetic (Level 4)	4+0x3	4+1x3	4+2x3	4+3x3	4+4x3 …	
Inductive stepHow many three?	(0)=1– 1	(1)=2 -1	(2)=3-1	(3)=4-1	(5)=6–1 …	(n–1)= n –1 This is just equal to the figure number minus 1 !
No. of matchsticks: Concluding (inducting) in arithmetic and algebra(Level 5)	4+ 3 (1– 1)	4 + 3 (2 -1)	4 + 3 (3-1)	4 + 3) (4-1	4 + 3 (6-1) …	4+3(n-1) = 1+3n = 3n +1

is the relation summarised by the function y= f(x) = 3x +1. From the fish diagram learners can deduce that for each value of x, there is a corresponding unique value of y.

The matchstick problem can be modelled in several ways (see Fig. 3) so that the same concept of a function that underlies the problem can be represented in various ways (Suh 2007) which representations on the surface and at first appear dissimilar. Multiple representations of concepts in these ways (Fig. 3) helps learners to see the links, the relationships between them that earlier seemed to be unrelated. Such multiple representations are the cornerstone of building conceptual understanding (Kilpatrick et al. 2001) of the function concept. This, as has been illustrated, builds from horizontal mathematisation when learners play with a story and begin see some patterns and move on to conjecture mathematical relations inherent. This leads to vertical mathematisation. When students have engaged in such learning they can always get back to horizontal mathematisation when they forget some aspects in vertical mathematisation because they will be familiar with the realistic contexts that motivate the formal mathematics.

Thus the matchstick pattern can be used to develop learners' conceptual understanding of the function concept using inductive reasoning through visuals. At each of the stage of multiple-representations, learners should be given ample time to discuss their deductions with their peers so that they can socially agree on the important skills described below that they are capable of developing:

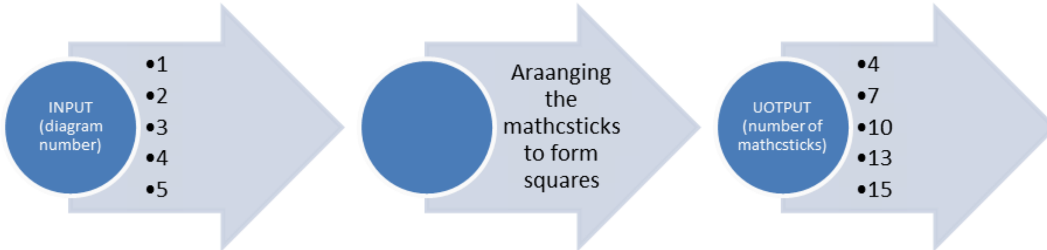

Fig. 3. Fish diagram for the matchstick problem

1. Verbally describing a pattern of figures built by squares formed with matchsticks
2. Visuals in the form of squares (see Fig.1)
3. The number pattern 4; 7; 10; 13; 16; …called an Arithmetic Progression, first term 1 and common difference 3.
4. A fish diagram (see Fig. 3)
5. Ordered pairs (1; 4), (2; 7), (3;10), (4;13),…
6. Diagrammatically as a pair of object - image sets which are matching (that is, those in one-one correspondence and have the same cardinality).
7. Algebraically as $y = 3x + 1$, or $f(x) = 3x + 1$, where x is a whole number (see Fig. 4).
8. Visually, much more formally in the form of a Cartesian graph.

The researcher believes that when number patterns are taught and learnt using the RME approach as illustrated above, mathematics can become sensible and motivating to learners. When developed using multiple representations arising from learners' contexts, the function concept may be fully understood. This can give learners a strong foundation for studying similar mathematical concepts successfully.

DISCUSSION

The power of the Realistic Mathematics Education approach in teaching different functional representations is that it bridges informal and formal mathematics. In particular it does not down-grade informal mathematics but rather, it uses it as a base for developing formal mathematics. Exploration of mathematical concepts using out-of-school, day-to-day approaches is

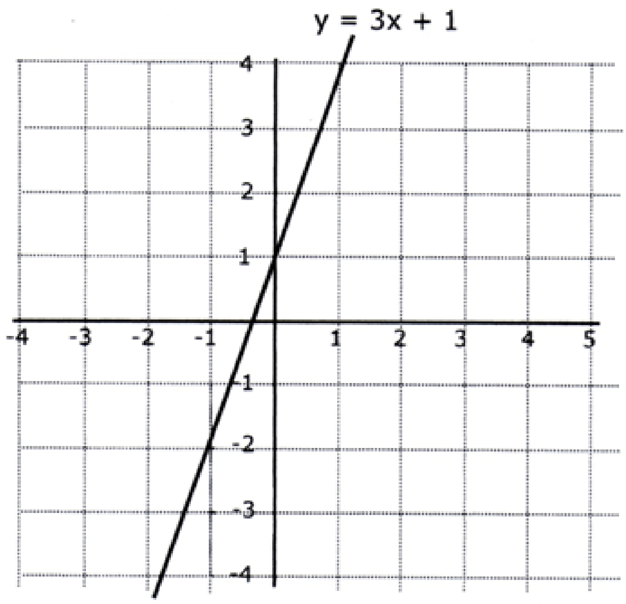

Fig. 4. Cartesian graph for the matchstick problem (vertical mathematisation)

encouraged as a basis of learning that can lead to strong understanding of mathematics concepts and processes. Such approaches help to break the everyday knowledge and school knowledge dichotomy which is a concern to many educational researchers (see Boaler 2008). When learners have problems with formal mathematics, such as in the use and meaning of mathematical formulae, they can always revert back to the first principles of horizontal mathematisation which gave rise to those formulae. The mechanistic (traditional) approach that emphasizes knowledge of mathematical symbols and mastery of procedures (Hiebert and Levevre 1986; Kilpatrick et al. 2001) is discounted by realistic approaches. The realistic approaches are grounded in mathematical contexts and conceptual knowledge (Hiebert and Levevre 1986) through horizontal mathematisation. Further the realistic mathematics education approaches as discussed in this paper have potential to raise learners' interest in mathematics as learners observe interrelationships in functional representations which at face value appear unconnected.

Different representations of functions through geometric patterns, number sequences (quadratic sequences, arithmetic sequences, geometric sequences or otherwise), discontinuous graphs on the Cartesian plane and the functional f(x) symbolism can help learners to gain insight of one of the biggest ideas in mathematics. As students move between different contexts and representations of the function concept they can realize that though mathematical ideas may appear different, that difference is often at a very superficial level. They need to see the difference between function and form. They can also come to realize that such differences in fact are sublimely inter-connected and interrelated by a single idea. Such perceptions are critical in raising learners' conceptual understanding, procedural fluency and productive disposition to mathematics (Kilpatrick et al. 2001). Learners who successfully learn through RME approaches begin to realise the logical structure and compactness of mathematics. They can see its explanatory power for daily problems and the underlying beauty of mathematics. Such advantages cannot be appreciated by those who regard mathematics with unsophistication. Once it dawns on students that mathematics is not a mechanical, capricious and mindless subject students can become more prepared and ready to devote extra effort to understand mathematics, even if at first they find it hard to understand a certain concept. Such learners are likely to regard anew mathematics concept from various perspectives in an effort to understand it. The realisation that mathematics is internally ordered, motivating and fun can only rise as students mathematise by making conjectures on real world phenomena that appeals to them.Then they would think of ways of solving can provide (Hallett and Bryant 2010; Siegler et al. 2013).

CONCLUSION

Regarding the teaching and learning of the function concept, the researcher argues that students can also understand that a sequence is a function whose domain is a set of natural numbers. Further, students will be able to notice that there are different types of functions; some continuous and some not. Some are formulae functions, some are trigonometric, and some are hyperbolic and so on. Later on at higher levels still, they can differentiate among injective, surjective and bijective functions. Such knowledge is the basis for studying linear algebra, abstract algebra, functional analysis and mathematical analysis for those who wish to specialise in mathematics at very high levels. The pertinent point being that once the basic notions of the function concept are studied properly at lower levels and the variations of the function concept are understood through building the concept through use of simple matchsticks as discussed in this paper for example, learners can be empowered to understand a big idea in mathematics and study mathematics meaningfully at any level.

The researcher strongly argues that if students study functions in the manner discussed in this paper, they can develop a strong foundation for mathematical conceptual and also mathematical procedural knowledge that helps them to study other mathematical concepts meaningfully. They can now see the need for special mathematical symbolisms and see that they are useful in generalizing their ideas as well for communicating their ideas to other people. RME helps learners to see the close relationship between mathematics conceptual knowledge and mathematical procedural knowledge. RME thus helps to diminish mathophobia and thus promotes productive disposition in mathematics which is the most important strand to promote meaningful learning of mathematics.

RECOMMENDATIONS

Given the argumentation based on the exposition of teaching the function concept in Grade 9 proposed in this paper, it is recommended that:

- the teaching of the function concept (and other mathematics concepts) must start with stories and games motivating and enchanting to learners in a way that the teacher can easily ask some questions that induce learners to think mathematically about the problem
- that teachers strive for learners to realise the multiple representations of a concept (stories, visuals, pictorials, graphical and then abstract) in their teaching to that learners develop a comprehensive understanding of the target mathematical concept
- that research be done to determine how the teaching of the functions topic using the RME and multiple representations approaches discussed in this paper would affect grade 9 learners' conceptual understanding of the topic.

The researcher is of the belief that if such recommendations are implemented in mathematics teaching learners' productive disposition of mathematics will greatly improve and their motivation for learning mathematics stoked up.

LIMITATIONS OF THE STUDY

This is a theoretical paper and therefore one of its weaknesses is that its conclusions have not been tested in the empirical field. However from this researcher's experience, many teachers use a procedural approach to teaching mathematics in which mathematical symbols and processes are foregrounded and taught in mathematics lessons as if for their own sake. This is often done without first exploring the contexts that motivate these processes. Also the researcher has observed that when mathematics is taught in this way learners and even teachers fail to see the big ideas that make mathematics such a powerful subject. Hence despite these limitations, the researcher believe this paper based on the Realistic Mathematics Education and use of multiple representations in mathematics teaching and learning proffers a fresh approach to teaching functions (and mathematics) in Southern Africa and beyond.

NOTE

This paper is written in memory of a dear colleague; the lae Prof. L.J. Nyaumue (born in Zimbabwe) who passed on in June 2012. He inspired me to work hard in my mathematics education research.

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